Friedrich-Alexander-Universität
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HYTEG
Hybrid Tetrahedral Grids

Multigrid in $\boldsymbol{H}(\mathbf{curl})$ on Hybrid Tetrahedral Grids

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Model Problem

- $\mathbf{curl}$-$\mathbf{curl}$ problem arising from Maxwell's equations [1]
- Relevant for e.g. nuclear fusion, semiconductor devices

Strong Form

$$\alpha\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + \beta\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega\,,$$
$$\boldsymbol{u}\times\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,.$$

Weak Form Find $\boldsymbol{u}\in \mathring{\boldsymbol{H}}(\mathbf{curl})$ st. $\forall \boldsymbol{v}\in\mathring{\boldsymbol{H}}(\mathbf{curl})$

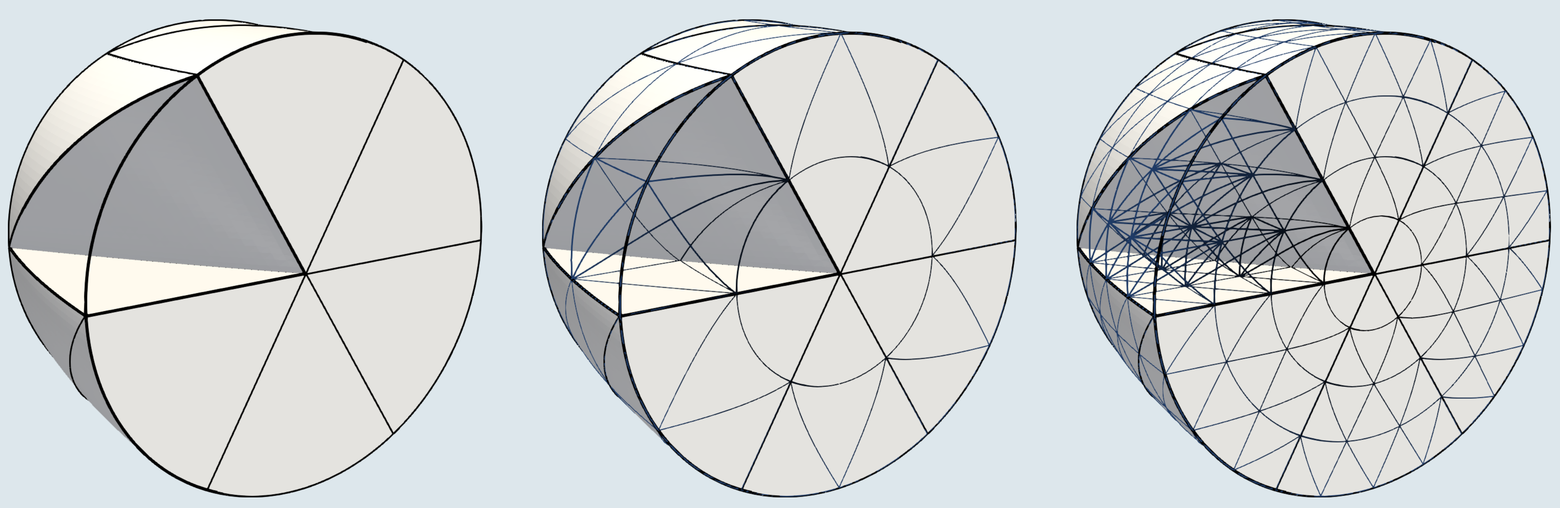
$$a(\boldsymbol{u},\boldsymbol{v}) := \alpha(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2(\Omega)} + \beta(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)} = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)}\,.$$

With $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$, $\alpha,\beta>0$, normal vector $\boldsymbol{n}$ and
$\mathring{\boldsymbol{H}}(\mathbf{curl}) := \{\,\boldsymbol{u}\in\boldsymbol{L}^2(\Omega) \mid \mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{L}^2(\Omega), \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \partial\Omega\,\}$.

HyTeG [2]

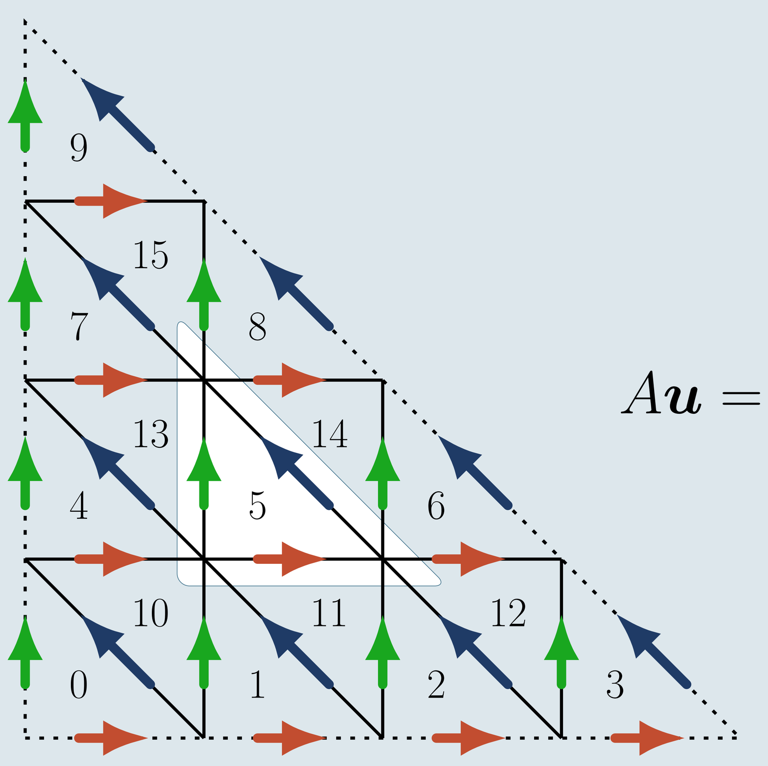
Hybrid Tetrahedral Grids

- Block-structured tetrahedral mesh: Flexible and efficient
- Mesh-hierarchy for multigrid
- Curvilinear on all levels



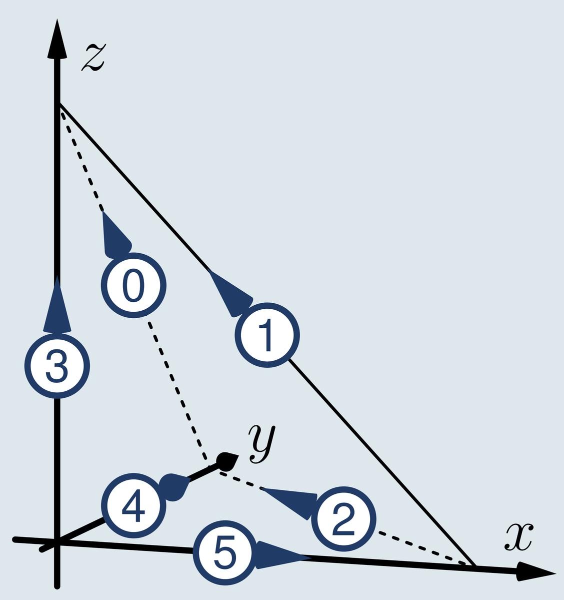
Matrix-free multigrid

- Storage of system matrix impossible at extreme scale ($1.1\cdot 10^{13}$ unknowns demonstrated in [3])
- Loop over elements
  1. Assemble element local matrix on-the-fly
  2. Perform local matrix-vector product
  3. Add local result to global vector



$$Au = \sum_{T\in\mathcal{T}} \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \int_T \phi_i\phi_j & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \rightarrow \\ \uparrow \\ \nwarrow \end{pmatrix}$$

Nédélec Elements (Type I, Order 1) $\mathcal{P}_1^-\Lambda^1$ [4, 5]



- Local space:
  $\{\,\boldsymbol{x}\mapsto\boldsymbol{a}\times\boldsymbol{x}+\boldsymbol{b},\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3\,\}$
- Degrees of freedom:
  $\boldsymbol{u}\mapsto\int_e \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{x}$
- Conforming in $\boldsymbol{H}(\mathbf{curl})$
- Continuous only in tangential direction

Hybrid Smoother [1]

Ellipticity

- Standard multigrid smoothers hinge on ellipticity
- $a(\cdot,\cdot)$ is not elliptic on $\mathcal{N}(\mathbf{curl})$
- Lifting to potential space recovers ellipticity

Structure Preserving Discretization

- Discrete Helmholtz decomposition

$$\mathcal{P}_1^-\Lambda^1(\mathcal{T}_h)\cap\mathcal{N}(\mathbf{curl}) = \mathbf{grad}\,\mathcal{P}_1^-\Lambda^0(\mathcal{T}_h)$$

- Commuting diagram

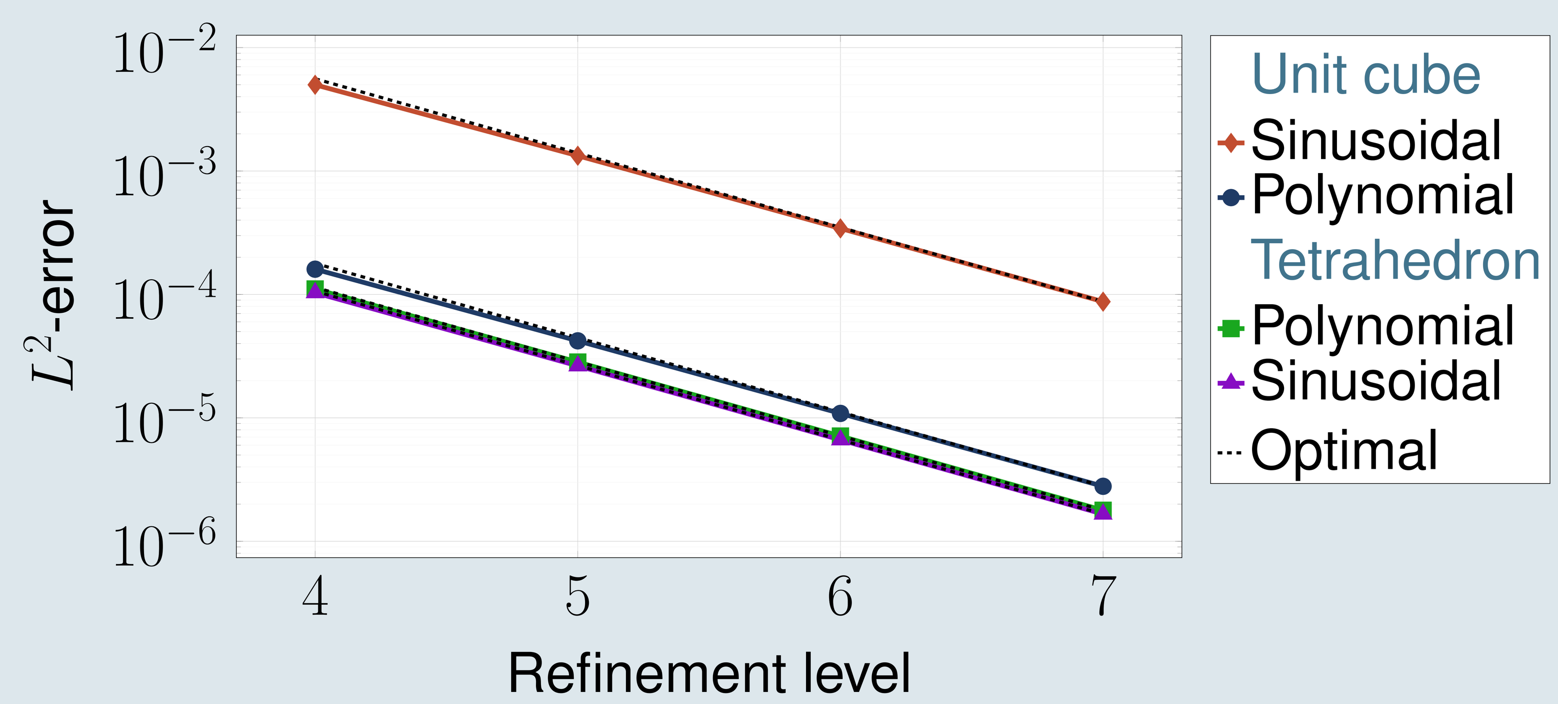
$$\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div};\Omega) \\
\downarrow{\scriptstyle \Pi^0} & & \downarrow{\scriptstyle \Pi^1} & & \downarrow{\scriptstyle \Pi^2} \\
\underbrace{\mathcal{P}_1^-\Lambda^0(\mathcal{T}_h)}_{\text{cont. piecew. lin.}} & \xrightarrow{\mathbf{grad}} & \underbrace{\mathcal{P}_1^-\Lambda^1(\mathcal{T}_h)}_{\text{Nédélec edge}} & \xrightarrow{\mathbf{curl}} & \underbrace{\mathcal{P}_1^-\Lambda^2(\mathcal{T}_h)}_{\text{Nédélec face}}
\end{array}$$

Input: initial guess $\boldsymbol{u}^h\in\mathcal{P}_1^-\Lambda^1$, right-hand side $\boldsymbol{f}^h\in\mathcal{P}_1^-\Lambda^1$
Output: smoothed iterate $\boldsymbol{u}^h$

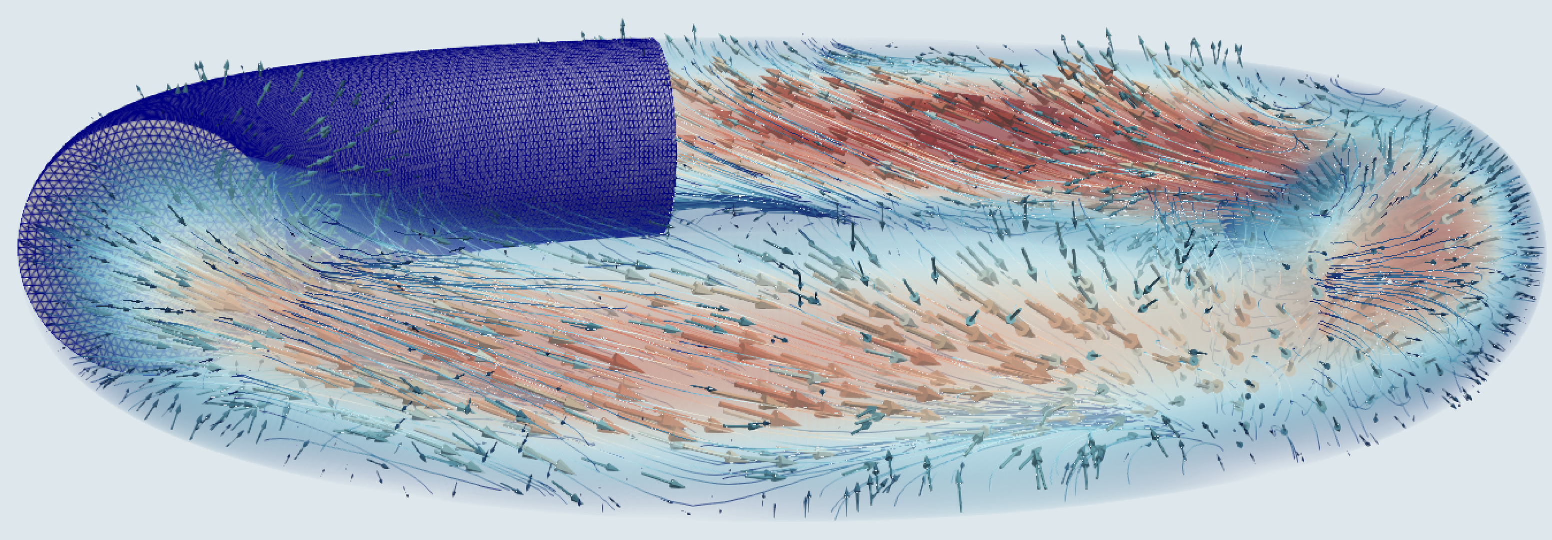
Relax on $A^h\boldsymbol{u}^h = \boldsymbol{f}^h$	`/* Smooth in` $\mathcal{N}(\mathbf{curl})^\perp$ `*/`
$\boldsymbol{r}^h \leftarrow \boldsymbol{f}^h - A^h\boldsymbol{u}^h$	`/* Calculate residual */`
$r^h \leftarrow L^h\boldsymbol{r}^h$	`/* Lift to potential space */`
$e^h \leftarrow 0$	
Relax on $\beta\,\Delta^h e^h = r^h$	`/* Smooth in` $\mathcal{N}(\mathbf{curl})$ `*/`
return $\boldsymbol{u}^h + L^{*h}e^h$	`/* Correct` $\boldsymbol{u}^h$ `*/`

Results

L^2 convergence



Solution to manufactured system on solid torus



References

[1] R. Hiptmair, "Multigrid method for Maxwell's equations,", 1998. DOI: 10.1137/S0036142997326203.
[2] N. Kohl, D. Thönnes, D. Drzisga, D. Bartuschat, and U. Rüde, "The HyTeG finite-element software framework for scalable multigrid solvers,", 2019. DOI: 10.1080/17445760.2018.1506453.
[3] B. Gmeiner, M. Huber, L. John, U. Rüde, and B. Wohlmuth, "A quantitative performance study for stokes solvers at the extreme scale,", 2016. DOI: 10.1016/j.jocs.2016.06.006.
[4] J. C. Nedelec, "Mixed finite elements in $\mathbb{R}^3$,", 1980. DOI: 10.1007/BF01396415.
[5] D. N. Arnold, *Finite Element Exterior Calculus*. 2018. DOI: 10.1137/1.9781611975543.